Linear systems – Final exam – Version A – Solutions

Final exam 2019–2020, Tuesday 16 June 2020, 8:30 – 12:00

Problem 1

(5 + 5 + 8 + 4 + 6 = 28 points)

A model for the spread of the Corona virus in a population is given as

$$\begin{aligned} \dot{s}(t) &= -\beta s(t)q(t), \\ \dot{q}(t) &= \beta s(t)q(t) - \gamma q(t), \\ \dot{r}(t) &= \gamma q(t), \end{aligned} \tag{1}$$

where $s(t) \in \mathbb{R}$, $q(t) \in \mathbb{R}$, and $r(t) \in \mathbb{R}$ are respectively the fraction of susceptible, infected, and recovered individuals. The parameter $\beta > 0$ represents the rate of infection, whereas $\gamma > 0$ is the rate of recovery.

Consider the initial value problem with dynamics (1) and initial condition $(s(0), q(0), r(0)) = (s_0, q_0, r_0)$ satisfying

$$s_0 + q_0 + r_0 = 1. \tag{2}$$

(a) To show that the corresponding solution satisfies $s(t) + q(t) + r(t) = 1$ for all $t \geq 0$, consider the variable

$$S(t) = s(t) + q(t) + r(t) \tag{3}$$

and note that

$$\dot{S}(t) = \dot{s}(t) + \dot{q}(t) + \dot{r}(t) = -\beta s(t)q(t) + \beta s(t)q(t) - \gamma q(t) + \gamma q(t) = 0, \tag{4}$$

as follows from (1). Hence, $S(\cdot)$ is a constant function of time. In addition, we have that $S(0) = s_0 + q_0 + r_0 = 1$, from which we can conclude that

$$S(t) = 1 \tag{5}$$

for all $t \geq 0$.

In the remainder of this problem, assume in addition to (2) that $s_0 > 0$, $q_0 > 0$, $r_0 \geq 0$, which can be shown to imply $s(t) > 0$, $q(t) > 0$, $r(t) \geq 0$ for all $t \geq 0$.

(b) Let $s(t) = S(r(t))$. Time-differentiation of this equality, using the chain rule for the right-hand side, leads to

$$\dot{s}(t) = \frac{\mathrm{d}S}{\mathrm{d}r}\dot{r}(t), \tag{6}$$

such that

$$\frac{\mathrm{d}S}{\mathrm{d}r} = \frac{\dot{s}(t)}{\dot{r}(t)} = -\frac{\beta}{\gamma}s(t) = -\frac{\beta}{\gamma}S. \tag{7}$$

Note that the latter result holds as we have assumed that $q(t) > 0$ for all $t \geq 0$, which implies that $\dot{r}(t) > 0$ for all $t \geq 0$ and the fraction is well-defined.

(c) To show that the fraction of susceptible and recovered individuals satisfies

$$s(t) = s_0 e^{-\frac{\beta}{\gamma}(r(t)-r_0)}.$$

for all $t \geq 0$, we consider (7). Note that (7) is a differential equation for S in which r is the independent variable. Then, separation of variables gives

$$\frac{1}{S}\,\mathrm{d}S = -\frac{\beta}{\gamma}\,\mathrm{d}r, \tag{8}$$

which leads to

$$\int \frac{1}{S}\,\mathrm{d}S = \int -\frac{\beta}{\gamma}\,\mathrm{d}r, \tag{9}$$

and, subsequently,

$$\ln|S| = -\frac{\beta}{\gamma}r + c \tag{10}$$

for some constant $c \in \mathbb{R}$. Next, note that $S(r(t)) = s(t) > 0$ by assumption on $s(\cdot)$, leading to

$$\ln|S| = \ln S = -\frac{\beta}{\gamma}r + c \tag{11}$$

and the general solution

$$S(r) = e^{\frac{\beta}{\gamma}r+c} = e^{c}e^{\frac{\beta}{\gamma}r}. \tag{12}$$

For $t = 0$, we have $S(r_0) = s_0$, such that

$$s_0 = e^{c}e^{-\frac{\beta}{\gamma}r_0} \tag{13}$$

leads to

$$e^{c} = s_0 e^{-\frac{\beta}{\gamma}r_0}. \tag{14}$$

Then, we have

$$S(r) = s_0 e^{-\frac{\beta}{\gamma}(r-r_0)}. \tag{15}$$

Substitution of $r = r(t)$ and $S(r(t)) = s(t)$ leads to the desired result.

(d) It can be shown that

$$\lim_{t\to\infty}\big(s(t), q(t), r(t)\big) = (s_\infty, 0, r_\infty),$$

for some s_∞, r_∞. To show that $(s_\infty, 0, r_\infty)$ is an equilibrium of the system (1), we substitute $(s, q, r) = (s_\infty, 0, r_\infty)$ in the right-hand side of (1) to obtain

$$\begin{aligned} \dot{s} &= 0, \\ \dot{q} &= 0, \\ \dot{r} &= 0, \end{aligned} \tag{16}$$

which is exactly the definition of an equilibrium. Note in addition that, by (a), we have $r_\infty + s_\infty = 1$

(e) To linearize the system (1) around the equilibrium $(s_\infty, 0, r_\infty)$, introduce the notation

$$x = \begin{bmatrix} s \\ q \\ r \end{bmatrix}, \qquad f(x) = \begin{bmatrix} -\beta x_1 x_2 \\ \beta x_1 x_2 - \gamma x_2 \\ \gamma x_2 \end{bmatrix} \tag{17}$$

as well as the equilibrium point

$$\bar{x} = \begin{bmatrix} s_\infty \\ 0 \\ r_\infty \end{bmatrix}. \tag{18}$$

Next, introduce the deviation from the equilibrium point as

$$\tilde{x}(t) = x(t) - \bar{x}, \tag{19}$$

whose dynamics we can approximate by linearization as

$$\dot{\tilde{x}}(t) = \frac{\partial f}{\partial x}(\bar{x})\tilde{x}(t). \tag{20}$$

A direct computation yields

$$\frac{\partial f}{\partial x}(x) = \begin{bmatrix} -\beta x_2 & -\beta x_1 & 0 \\ \beta x_2 & \beta x_1 - \gamma & 0 \\ 0 & \gamma & 0 \end{bmatrix}, \tag{21}$$

such that

$$\frac{\partial f}{\partial x}(\bar{x}) = \begin{bmatrix} 0 & -\beta s_\infty & 0 \\ 0 & \beta s_\infty - \gamma & 0 \\ 0 & \gamma & 0 \end{bmatrix}. \tag{22}$$

Problem 2 (14 points)

Consider the linear system $\dot{x}(t) = Ax(t)$ with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -b & -b & -3a & -a \end{bmatrix}$$

where $a, b \in \mathbb{R}$. In order to determine the values of a and b for which the system is asymptotically stable, note that A is in so-called companion form. As a result, its characteristic polynomial is immediately obtained as

$$\Delta_A(s) = s^4 + as^3 + 3as^2 + bs + b. \tag{23}$$

Next, we note that the system is asymptotically stable if and only if the polynomial (23) is stable. Therefore, we will verify stability of (23) using the Routh-Hurwitz test.

In particular, this leads to the following table:

		s^4	s^3	s^2	s^1	s^0	
$a\times$	Δ_A	1	a	$3a$	b	b	(step 0)
$1\times$		a		b			
$(3a^2-b)\times$	q		a^2	$3a^2-b$	ab	ab	(step 1)
$a^2\times$			$3a^2-b$		ab		
	r			$(3a^2-b)^2$	$(2a^2-b)ab$	$(3a^2-b)ab$	(step 2)

Note that a necessary condition for stability of a polynomial is that all coefficients have the same sign (and are nonzero). Thus, it immediately follows from Δ_A (step 0) that

$$a > 0, \qquad b > 0 \tag{24}$$

are necessary for stability. We assume (24) from now on.

Note that the first two leading coefficients of Δ_A are nonzero and have the same sign (from (24)), such that the Routh-Hurwitz criterion leads to the polynomial q as a result of step 1. As we have that $a > 0$ and $b > 0$, a necessary condition for stability is that

$$3a^2 - b > 0. \tag{25}$$

Assuming (24) and (25), we note that the leading coefficients of q are nonzero and have the same sign, such that we can do a second step to obtain the polynomial r. This leads to the necessary condition

$$2a^2 - b > 0, \tag{26}$$

which can be seen to imply (25).

Note now that r is a second-order polynomial, which is known to be stable if and only if all coefficients are nonzero and have the same sign. This is the case when (24) and (26) hold, such that (24) and (26) are both necessary and sufficient conditions.

By the Routh-Hurwitz criterion, this implies that Δ_A is stable (and the system asymptotically stable) if and only if

$$0 < a, \quad 0 < b < 2a^2. \tag{27}$$

Problem 3 (4 + 12 + 8 = 24 points)

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t),$$

with $x(t) \in \mathbb{R}^2$, $y(t) \in \mathbb{R}$, and where

$$A = \begin{bmatrix} 2 & 6 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} -1 & -4 \end{bmatrix}.$$

(a) To verify observability, compute

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ -2 & -10 \end{bmatrix}, \tag{28}$$

such that

$$\operatorname{rank} \begin{bmatrix} C \\ CA \end{bmatrix} = \operatorname{rank} \begin{bmatrix} -1 & -4 \\ -2 & -10 \end{bmatrix} = 2, \tag{29}$$

i.e., the system is observable.

(b) We aim to find a nonsingular matrix T and real scalars α_1, α_2 such that

$$TAT^{-1} = \begin{bmatrix} 0 & \alpha_1 \\ 1 & \alpha_2 \end{bmatrix}, \qquad CT^{-1} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Transposing the desired result leads to

$$(TAT^{-1})^{\mathrm{T}} = T^{-\mathrm{T}}A^{\mathrm{T}}T^{\mathrm{T}} = \begin{bmatrix} 0 & 1 \\ \alpha_1 & \alpha_2 \end{bmatrix}, \qquad (CT^{-1}) = T^{-\mathrm{T}}C^{\mathrm{T}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{30}$$

We note that the matrix pair (A, C) is observable if and only if the matrix pair $(A^{\mathrm{T}}, C^{\mathrm{T}})$ is controllable. In addition, the matrices in (30) are in the controllability canonical form, such that we can apply the algorithm from Theorem 4.13.

As a first step, compute the characteristic polynomial

$$\Delta_A(s) = \Delta_{A^{\mathrm{T}}}(s) = \det(sI - A) = \begin{vmatrix} s-2 & -6 \\ 0 & s-1 \end{vmatrix} = (s-2)(s-1) = s^2 - 3s + 2, \tag{31}$$

which is of the form

$$\Delta_A(s) = s^2 + a_1 s + a_0 \tag{32}$$

with $a_1 = -3$ and $a_0 = 2$. This immediately leads to

$$\alpha_1 = -a_0 = -2, \qquad \alpha_2 = -a_1 = 3. \tag{33}$$

To find the corresponding transformation T, compute

$$q_2 = C^{\mathrm{T}} = \begin{bmatrix} -1 \\ -4 \end{bmatrix} \tag{34}$$

and

$$q_1 = A^{\mathrm{T}}C^{\mathrm{T}} + a_1 C^{\mathrm{T}} = \begin{bmatrix} -2 \\ -10 \end{bmatrix} - 3\begin{bmatrix} -1 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \tag{35}$$

Now, we can define T^{T} as

$$T^{\mathrm{T}} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & -4 \end{bmatrix} \tag{36}$$

such that

$$T = \begin{bmatrix} 1 & 2 \\ -1 & -4 \end{bmatrix}. \tag{37}$$

(c) To design a stable state observer of the form

$$\dot{\xi}(t) = A\xi(t) + Bu(t) + G(y(t) - C\xi(t)),$$

we need to find G such that $A - GC$ has eigenvalues -2 and -3.

To this end, we define the desired polynomial as the polynomial p whose roots are the desired eigenvalues, i.e.,

$$p(s) = (s+2)(s+3) = s^2 + 5s + 6 = s^2 + p_1 s + p_0, \tag{38}$$

with $p_0 = 6$ and $p_1 = 5$. Next, observe that

$$\Delta_{A-GC}(s) = \Delta_{T(A-GC)T^{-1}}(s), \tag{39}$$

such that we can perform the design using the matrices in the canonical form. After defining

$$TG = \begin{bmatrix} g_0 \\ g_1 \end{bmatrix}, \tag{40}$$

we obtain

$$T(A-GC)T^{-1} = TAT^{-1} - TGCT^{-1} = \begin{bmatrix} 0 & -a_0 - g_0 \\ 1 & -a_1 - g_1 \end{bmatrix}, \tag{41}$$

which is in companion form and therefore has the characteristic polynomial

$$\Delta_{T(A-GC)T^{-1}}(s) = s^2 + (a_1 + p_1)s + (a_0 + p_0). \tag{42}$$

A comparison of (42) with the desired polynomial (38) leads to

$$a_0 + g_0 = 6, \qquad a_1 + g_1 = 5, \tag{43}$$

which can be solved to obtain

$$g_0 = 6 - a_0 = 4, \qquad g_1 = 5 - a_1 = 8. \tag{44}$$

Solving the linear system (40) with T as in (37), e.g., by using the matrix inverse

$$T^{-1} = \frac{1}{2}\begin{bmatrix} 4 & 2 \\ -1 & -1 \end{bmatrix}, \tag{45}$$

leads to

$$G = \begin{bmatrix} 16 \\ -6 \end{bmatrix}. \tag{46}$$

Problem 4 (5 + 5 = 10 points)

Consider the linear system

$$\dot{x}(t) = \begin{bmatrix} -1 & -4 & 0 \\ 0 & 3 & 0 \\ 6 & 13 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} u(t).$$

(a) To determine whether the system is controllable, denote

$$A = \begin{bmatrix} -1 & -4 & 0 \\ 0 & 3 & 0 \\ 6 & 13 & -4 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \tag{47}$$

and compute

$$\begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & -4 & -8 \\ 1 & 3 & 9 \\ 3 & 1 & 11 \end{bmatrix}. \tag{48}$$

It is clear that the first two columns are linearly independent, but note that the third column is a linear combination of the first two columns (take two times the second column plus three times the first column). Hence, we have

$$\operatorname{rank} \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \operatorname{rank} \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 0 & -4 & -8 \\ 1 & 3 & 9 \\ 3 & 1 & 11 \end{bmatrix} = 2 < 3, \tag{49}$$

and the system is not controllable. The reachable subspace is given as

$$\begin{aligned} \mathcal{W} = \operatorname{im} \begin{bmatrix} 0 & -4 & -8 \\ 1 & 3 & 9 \\ 3 & 1 & 11 \end{bmatrix} = \operatorname{span}\{ \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix} \} &= \operatorname{span}\{ \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -8 \end{bmatrix} \} \\ &= \operatorname{span}\{ \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \} \end{aligned} \tag{50}$$

such that a basis is given by the vectors

$$\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}. \tag{51}$$

(b) The system is stabilizable if

$$\operatorname{rank} \begin{bmatrix} A - \lambda I & B \end{bmatrix} = n, \qquad \forall \lambda \in \sigma(A) \text{ s.t. } \Re(\lambda) \geq 0. \tag{52}$$

From the lower block triangular structure of A in (47), it follows that

$$\sigma(A) = \{-4\} \cup \sigma\left(\begin{bmatrix} -1 & -4 \\ 0 & 3 \end{bmatrix} \right) = \{-4, -1, 3\}, \tag{53}$$

where the latter equality follows as the upper left block of A is itself upper triangular. Hence, we only need to verify (52) for $\lambda = 3$, which leads to

$$\begin{bmatrix} A - 3I & B \end{bmatrix} = \begin{bmatrix} -4 & -4 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 6 & 13 & -7 & 3 \end{bmatrix}. \tag{54}$$

This resulting matrix is easily seen to have rank three, i.e., the system is stabilizable.

Problem 5 (14 points)

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t), \tag{55}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$.

We consider its reachable subspace

$$\mathcal{W} = \operatorname{im} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}, \tag{56}$$

as well as its unobservable subspace

$$\mathcal{N} = \ker \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}. \tag{57}$$

It is assumed that $\mathcal{W} \subset \mathcal{N}$ and we aim to prove that this implies $H(t) = 0$ for all $t \in \mathbb{R}$, where H is the impulse response matrix given by

$$H(t) = \begin{cases} Ce^{At}B \;, & t \geq 0 \\ 0 \;, & t < 0 \end{cases} \tag{58}$$

As a result, it is sufficient to show that

$$Ce^{At}B = 0 \tag{59}$$

for all $t \geq 0$.

Let $x \in \mathcal{W}$, i.e., there exists a vector $r \in \mathbb{R}^{nm}$ such that

$$x = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} r. \tag{60}$$

As $\mathcal{W} \subset \mathcal{N}$, we also have $x \in \mathcal{N}$, which implies

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} r = 0. \tag{61}$$

In fact, as $x \in \mathcal{W}$ can be taken arbitrarily, the above holds for any vector $r \in \mathbb{R}^{nm}$, which implies that

$$CA^kB = 0 \tag{62}$$

for $k = 0, 1, 2, \ldots, 2n-2$. By the theorem of Cayley-Hamilton, this leads to

$$CA^kB = 0 \tag{63}$$

for $k = 0, 1, 2, \ldots$.

Next, we consider (59), which can be written as

$$Ce^{At}B = C\left(\sum_{k=0}^{\infty} \frac{A^k t^k}{k!}\right) B = \sum_{k=0}^{\infty} \frac{CA^kBt^k}{k!} \tag{64}$$

by using the definition of the matrix exponential. Clearly, we have

$$Ce^{At}B = 0 \tag{65}$$

due to (63), which proves the desired result.